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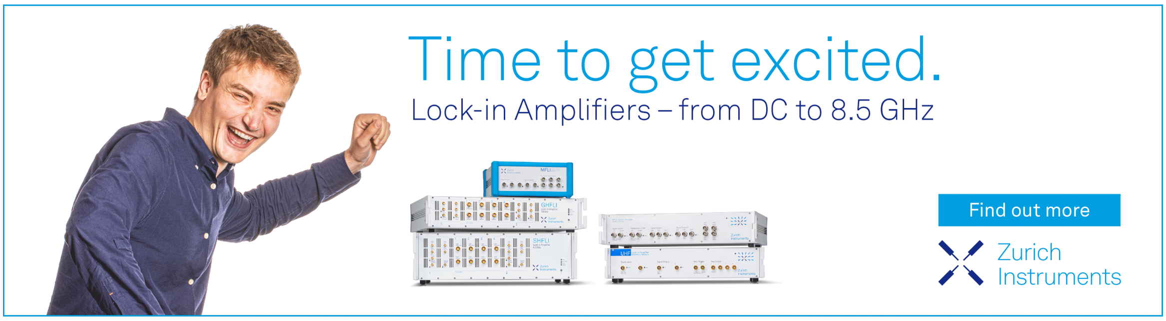

Dynamical Analysis of Massless Charged Particles

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Abstract. We present an 8-dimensional dynamical framework that describes the behavior of charged particles of arbitrary mass in the presence of general external electromagnetic fields. In particular, we discuss the applicability of this framework to the massless sector of the theory. Relativistic systems are known to have constrained dynamics, and the proposed scheme can handle this kind of dynamics. We also present a prescription to properly select the initial conditions compatible with the constrained dynamics.

INTRODUCTION

Although no experimental evidence has been found to indicate the existence of fundamental massless charged particles, there are some physical scenarios, such as the graphene conduction electrons [1, 2], the semi-Weyl metals [3], or the two-level non-Hermitian quantum systems [4], among others, in which an effective description of massless charged quasiparticles emerges. In relativistic theory, massless particles propagate at the light speed describing null geodesics; on the other hand, if the particles also have an electric charge, they deviate from the geodesics mentioned above due to the interaction with external electromagnetic fields but maintaining the magnitude of their velocity c. This behavior characterizes massless charged particles [5].

There is extensive specialized literature devoted to the study of the classical dynamics of electrically charged particles, with or without spin, massive or not (see [5], and references therein for a review on this topic). In this work, we restrict ourselves to the Lagrangian formalism developed in [6, 7, 8], which naturally allows the transition to the Hamiltonian formalism, which in turn establishes the basis for canonical quantization [9]. In the context of nonlinear dynamics, the behavior of charged particles in the presence of electromagnetic fields represents an interesting case study in its own right. For example, the trajectories of massive charged particles in the presence of a magnetic dipole field can be periodic, quasiperiodic, chaotic, and even hyperchaotic; both in the nonrelativistic [10] and in the relativistic [11, 12] regimes. Based on the generalization proposed in [5], in principle, these results can be extended to the null mass sector of the charged particle theory. It is desirable to obtain a dynamical system represented by a set of first-order differential equations in this context. However, there are some algebraic equations that link some degrees of freedom in the phase space, therefore the dynamics of relativistic particles is constrained. The constraints must be considered together with the dynamic equations.

When applying a numerical integration scheme to a constrained dynamic system, it is essential to keep in mind that: (i) the initial conditions of the system must be compatible with the constraints and (ii) the constraint equations must remain valid - within a value of given tolerance - after each iteration. This last aspect deserves careful treatment since a numerical integrator contains an error associated with the numerical precision of the calculation; in turn, this leads to the possibility that, after a specific integration time, the system leaves the constraint hypersurface. When this situation occurs, the numerical solutions provided by the integrator will no longer correspond to the set of physically acceptable solutions and might diverge. This work aims to develop a general dynamic scheme that allows the study of charged particles, relativistic or not, and with arbitrary mass (i.e., massive or massless), and a layout including the presence of constraints.

CLASSICAL THEORY OF CHARGED PARTICLES OF ARBITRARY MASS

In what follows, we adopt the Minkowski's metric with signature $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and the natural units $c = 1$; the space-time coordinates denoted by $x^0 = t$, $\mathbf{x} = \{x,y,z\}$; and finally, Einstein's summation convention of repeating indices. The action for the relativistic particle of mass m and charge q coupled to an external electromagnetic

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field $A_\mu(x)$ is given by [5, 6, 7, 8]:

$$S = -\int_{\mathscr{C}} d\lambda e(\lambda)\left(\frac{1}{2}\frac{\dot{x}(\lambda)^2}{e(\lambda)^2} + \frac{1}{2}m^2 + qA_\mu(x(\lambda))\frac{\dot{x}^\mu(\lambda)}{e(\lambda)}\right), \tag{1}$$

where λ represents a parameterization of the world line, $\mathscr{C}$, described by the particle; derivatives with respect to this parameter are represented by a dot over the variable under consideration. The action also contains an additional degree of freedom, $e(\lambda)$, necessary to ensure invariance under general reparameterizations [13]. Note that the mass term is decoupled from the kinetic term so that the sector $m = 0$, is well defined.

The variation of the action, Eq. (1), with respect to x leads to the relativistic version of Newton's second law:

$$\frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{e}\right) = qF_{\mu\nu}\dot{x}^\nu, \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor. On the other hand, the variation with respect to e gives as a result

$$\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2e^2 = 0. \tag{3}$$

This equation is not a dynamical one but represents a constraint in the space of the 4-velocities. When $m \neq 0$ it is possible to reduce the space of physical solutions as follows: the solution of the constraint leads to writing down $e = \sqrt{\dot{x}^2}/m$, expression that is then substituted in Eq. (2), from which, we obtain the well-known set of equations describing the dynamics of massive charged particles.

The situation is qualitatively different when $m = 0$. Thus, the constraint reduces to $\dot{x}^2 = 0$. Consequently, the particle propagates now at the speed of light. Note that the constraint does not provide any information about e in this case. It is instructive to set forth the dynamical equations for this case, adopting the parameterization $\lambda = t$, being t the time measured in the laboratory frame. Accordingly, we have [14]

$$\frac{d}{dt}\left(\frac{1}{e(t)}\right) = q\mathbf{E}\cdot\mathbf{v}, \tag{4}$$

$$\frac{d}{dt}\left(\frac{\mathbf{v}}{e(t)}\right) = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right), \tag{5}$$

along with the algebraic equation

$$\dot{\mathbf{x}}^2 - 1 = 0. \tag{6}$$

This system contains four differential equations, one of the first order and three of the second order. This fact suggests we require seven integration constants; however, the constraint equation eliminates one of the velocity components. Thus, the number of independent initial conditions is six. We see that the essential difference existing between the massive and massless cases is that in the first case, the velocity components are independent of each other (but with the velocity module being bounded by the inequality $|\mathbf{v}| < 1$), while in the second case only two components are independent.

A GENERALIZED DYNAMICAL FRAMEWORK

In order to get a unified dynamical framework that allows us to study the behavior of charged particles of arbitrary mass (including the massless case) we proceed as follows. Let us introduce an 8–dimensional phase space with coordinates defined as [15] [16]

$$\{t, \mathbf{x}, \mathscr{E}, \mathbf{p}\} \equiv \left\{t, \mathbf{x}, \frac{\dot{t}}{e}, \frac{\dot{x}}{e}\right\}. \tag{7}$$

In this space the constraint, Eq. (3), is written as

$$\phi \equiv \mathscr{E}^2 - \mathbf{p}^2 - m^2 = 0. \tag{8}$$

We recognize in this equation the well known *dispersion relation* that defines hyperbolas in 4-momenta space, with vertex localized at $\mathscr{E} = \pm m$; for massless particles the hyperbolas degenerate in a double cone. In terms of the variables defined by Eq. (7), Eq. (2) reads as

$$\dot{t} = f, \tag{9}$$

$$\dot{\mathbf{x}} = \frac{f}{\mathscr{E}}\mathbf{p}, \tag{10}$$

$$\dot{\mathscr{E}} = q\frac{f}{\mathscr{E}}\mathbf{p}\cdot\mathbf{E}, \tag{11}$$

$$\dot{\mathbf{p}} = q\frac{f}{\mathscr{E}}\left(\mathscr{E}\mathbf{E} + \mathbf{p}\times\mathbf{B}\right), \tag{12}$$

where we introduced the non-physical arbitrary function $f = f(\lambda)$ which needs to be fixed by a particular parameterization. The promotion of t to the status of a dynamical variable by the introduction of f ensures the system to be autonomous [17], even when the external electromagnetic field is time dependent. Just to mention a couple of examples, one possible parameterization is $f = \mathscr{E}(\lambda)$; other possibility is $f = \omega$, with ω a constant. In any case, the choice of a particular parameterization will not change the physical content of the theory.

The parameterization $f = 1$ corresponds to the time measured from a laboratory frame of reference, as it can be seen by solving Eq. (9). For massive particles, Eqs. (7) and (3) imply that $\mathscr{E} = \gamma(\dot{\mathbf{x}})$, where $\gamma = (1-\dot{\mathbf{x}}^2)^{-1/2}$ is the Lorentz factor. Even though we know that this factor has an explicit dependence on velocity, we will keep it as an independent variable of the 8–dimensional phase space. This is a crucial step in order to extend the formalism to massless particles, as for this case particles propagate at the speed of light, hence the identification $\mathscr{E} = \gamma(c) = \infty$ is no longer valid. The physical interpretation of $\mathscr{E}$ is that of the kinetic energy plus the rest mass [5].

Last but not least, the dynamical system must be consistent with the constraint, Eq. (8) in the following sense: if $\phi(t_0) = 0$, then $\phi(t) = 0$ for $t > t_0$. In other words, the time derivative of the original constraint, $\dot{\phi} = 2\left(\mathscr{E}\dot{\mathscr{E}} - \mathbf{p}\cdot\dot{\mathbf{p}}\right) = 0$, must be also a constraint (a *secondary* one). We use Eq. (12) to eliminate $\dot{\mathbf{p}}$ from the secondary constraint, from which we obtain Eq. (11). This result is telling us that it would suffice to choose a set of initial conditions compatible with the constraint at a given time, since from that instant the system will evolve fulfilling the original constraint.

DETERMINING THE SET OF INITIAL CONDITIONS AND THE VELOCITY SPHERE

The velocity $\dot{\mathbf{x}}$ is not invariant under reparameterizations, hence it does not represent a physical quantity. The physical velocity is defined as $\mathbf{v} = \dot{\mathbf{x}}/\dot{t}$. From Eqs. (9) and (10) we can check that $\mathbf{v} = \mathbf{p}/\mathscr{E}$, a well-known result from the theory of relativity. We note that given an orbit in the 8–dimensional phase space, it is always possible to reconstruct the physical velocity; this is valid for arbitrary values of mass, and by construction, invariant under reparameterizations. When we select an appropriate set of initial conditions compatible with the constraint, we are also fixing the initial velocity of the particle since $\mathbf{v}(0) = \mathbf{p}(0)/\mathscr{E}(0)$. In order to be more specific with our discussion, in what follows, let us adopt the parameterization $\lambda = t$, so from now on, we can indistinctly refer to the velocity as $\dot{\mathbf{x}}$ or $\mathbf{v}$.

We define the initial state of the system as

$$\{t(0), \mathbf{x}(0), \mathscr{E}(0), \mathbf{p}(0)\} = \{t_0, \mathbf{x}_0, \mathscr{E}_0, \mathbf{p}_0\}.$$

The initial position must be chosen from the domain $\mathbf{x}_0 \in \mathbb{R}^3$. On the other hand, $\mathscr{E}_0$ and $\mathbf{p}_0$ are not independent because of the constraint equation. We set the parameter m to a given value, then $\mathscr{E}_0 \geq m$, and consequently the admissible initial conditions for $\mathbf{p}$ must be such that $\mathbf{p}_0^2 = \mathscr{E}_0^2 - m^2$, which is the equation of a sphere of radius $(\mathscr{E}_0^2 - m^2)^{1/2}$ defined in tridimensional momenta space. One additional simplification can be achieved if we introduce the dimensionless parameter $\mu \equiv m/\mathscr{E}_0$, thus $0 \leq \mu \leq 1$. This definition allows us to consider the mass of the particle as a discrete index, with values $m = 0, 1$, for massless and massive particles, respectively. It would be useful to reconsider the initial velocity, $\mathbf{v}_0$, instead of the initial momenta. The main reason for this is that velocity space is bounded by the physical limit $|\mathbf{v}| \leq 1$, whereas momenta space is not, in fact, $\mathbf{p}_0 \in \mathbb{R}^3$. Just as we pointed out before, once we solve the dynamical system, it is always possible to reconstruct the corresponding orbits in velocity space. For massive particles, it is satisfied

$$\mathbf{v}^2(t) = 1 - \left(\frac{m}{\mathscr{E}(t)}\right)^2, \tag{13}$$

thus a generic orbit in velocity space will be contained within the interior of a unitary sphere (dubbed as the *velocity sphere*). In contrast, the distinctive property of massless particles is that their orbits cannot explore the interior of the velocity sphere, they are restricted to lay on its surface, this in concordance with the fact that they propagate at the speed of light all the time. For $t = 0$, Eq. (13) implies that only two components of the velocity are independent. For instance, we may choose some numerical values for v_{0x} and v_{0z}, accordingly $v_{0y} = \pm\sqrt{1-\mu^2 - v_{0x}{}^2 - v_{0z}{}^2}$. This procedure requires extra care because the chosen components v_{0x}, v_{0z} may lead to imaginary values for v_{0y}, not to mention that we also need to set by hand the sign of v_{0y}. As an alternative to overpass these difficulties, let us introduce spherical coordinates in velocity space:

$$\begin{aligned} v_{0x} &= \sqrt{1-\mu^2}\sin\psi_0\cos\phi_0, \\ v_{0y} &= \sqrt{1-\mu^2}\sin\psi_0\sin\phi_0, \\ v_{0z} &= \sqrt{1-\mu^2}\cos\psi_0, \end{aligned}$$

where $(\psi_0, \phi_0) \in [-\pi/2, \pi/2] \times [-\pi, \pi]$, represent the latitude and longitude, respectively, of a point localized on the surface of a sphere of radius $\sqrt{1-\mu^2}$. It is clear that in this representation only two independent numbers are needed to fulfill the conditions for the initial velocity of the particle. It is straightforward to translate these two numbers to initial conditions in momenta space since we already know that $\mathbf{p}_0 = \mathscr{E}_0\mathbf{v}_0$.

To summarize, the 6-dimensional space, denoted by $\mathscr{S}$, that defines the set of initial conditions compatible with the constraint, Eq. (8), is defined by

$$\begin{aligned} \mathscr{E}(0) &= \mathscr{E}_0 \in [m, \infty), \quad m = 0, 1, \\ \mathbf{x}(0) &= \mathbf{x}_0 \in \mathbb{R}^3, \\ \mathbf{p}(0) &= \mathscr{E}_0\mathbf{v}_0, \quad \mathbf{v}_0 \in S^2, \end{aligned}$$

or equivalently, $\mathscr{S} = \mathbb{R}^3 \times [m, \infty] \times S^2$.

CONCLUSIONS

Based on Ref. [5], we have proposed a general dynamical scheme that allows us to describe the dynamics of relativistic and nonrelativistic charged particles, massive or massless, and in the presence of arbitrary external electromagnetic fields. Generally speaking, the dynamics of relativistic systems is constrained. This situation represents a potentially dangerous problem for dynamical systems that exhibit sensitivity to initial conditions, since intrinsic errors due to numerical integration are likely to be amplified, leading to orbits that leave the constraint hypersurface in phase space, and consequently leading to nonphysical solutions.

The program proposed in this work promotes time and energy as new independent degrees of freedom, giving rise to an 8-dimensional phase space. For massive particles, this is equivalent to promoting the Lorentz factor (the coordinate $\mathscr{E}$) to the status of a dynamical variable. This factor is not well defined for massless particles, but once we considered it as an independent degree of freedom, the framework adapts quite straightforwardly to this situation. Furthermore, this framework is able to deal with the constrained dynamics due to the fact that the dynamical equation for $\mathscr{E}$ is no other thing than a consistency requirement for the original constraint to be preserved in time. From a numerically integration point of view, it should be sufficient to satisfy the constraint by choosing a compatible set of initial conditions, so the set of dynamical equations will guarantee the validity of the constraint as the system evolves in time.

The dynamical framework includes a prescription to choose a suitable set of initial conditions. To start with, the mass of the particle is reintroduced as a normalized discrete index, $m = 0, 1$ for the massless and the massive sector, respectively. The constrained dynamics imply that only two components of the initial velocity are independent. In this regard, we have introduced the so-called velocity sphere: a 2-dimensional space defined as the surface of a sphere with radius $\sqrt{1-\mu^2}$, with $0 \leq \mu \leq 1$. The case $\mu = 0$ corresponds to massless particles. The introduction of the velocity sphere manifestly shows that only two numbers need to be chosen, i.e., longitude and latitude, which fully define the particle's initial velocity.

A dynamical framework that includes the case of massless particles could well be useful to describe the behavior of ultra-relativistic particles, for which it has been evidenced that large Lorentz factors represent a challenge for numerical integration [18]. Other possible application is the study of the dynamics of photons interacting with strong gravitational fields. Finally, we want to point out the possibility to include intrinsic spin degrees of freedom through the introduction of Grassmannian coordinates [5, 6, 7, 8], this can be also accomplished by an extra coupling with torsion of the world line of the particle [19, 20, 21]. It remains open the possibility to extend the present dynamical framework to the aforementioned theoretical scenarios.

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13. That is to say $\lambda \to \lambda' = \lambda'(\lambda)$, such that the one dimensional volume element, $e(\lambda')d\lambda' = e(\lambda)d\lambda$, remains invariant.
14. We define $\mathbf{E} = \{F_{01}, F_{02}, F_{03}\}$ and $\mathbf{B} = \{-F_{23}, -F_{31}, -F_{12}\}$.
15. This must not be confused with the canonical phase space, for which $\{x^\mu, \pi_\nu\} = \delta^\mu_\nu$, with $\pi_\mu \equiv -p_\mu - qA_\mu$.
16. As mentioned before (see explanation following Eq. (1)), we want to emphasize that the evolution parameter is λ, and thus $\dot{t} = dx^0(\lambda)/d\lambda$ stands for the velocity of the time coordinate.
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